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Slow evolution of nearly-degenerate extremal surfaces

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Abstract

It was conjectured recently that the string worldsheet theory for the fast moving string in AdS times a sphere becomes effectively first order in the time derivative and describes the continuous limit of an integrable spin chain. In this paper we will try to make this statement more precise. We interpret the first order theory as describing the long term evolution of the tensionless string perturbed by a small tension. The long term evolution is a Hamiltonian flow on the moduli space of periodic trajectories. It should correspond to the renormgroup flow on the field theory side.

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1. Introduction

The AdS/CFT correspondence relates weakly coupling limit of the Type IIB string theory to the strongly coupled limit of the $\mathcal{N} = 4$ Yang–Mills theory. It is hard to imagine that this type of a correspondence would allow for quantitative checks besides the comparison of the quantities protected by the supersymmetry. But the recent research revealed several examples where some nontrivial parts of the Yang–Mills perturbation theory are reproduced from the string theory computations. The first work in this direction was the computation of the expectation value of the circular Wilson loop [1,2]. It was followed by the discovery of the BMN limit [3–5] and the "spinning string" solutions which we will discuss in this paper. In these computations supersymmetry alone is not enough to guarantee the agreement of the results of the string theory and the field theory. It turns out that in some field theory computations the perturbation series depend on the coupling constant λ only in the combination λ/J^2 where J is a large integer. If $J^2 \gg \lambda$ the perturbative computations can presumably be trusted even when λ is large, and when λ is large they can be matched with the string theory computations. At the moment there is no solid explanation of why it works, and even whether this is true to all orders of the Yang–Mills perturbation theory (see [6] for one of the most recent discussions.) But there are several computations with the impressive agreement between the field theory and the string theory.

The "spinning string" solutions were first considered in the context of the AdS/CFT correspondence in [7–9]. Various computations in the classical dynamics of these solutions lead to the series in the small parameter which on the field theory side is identified with λ/J^2. It was conjectured that the Yang–Mills perturbation theory in the corresponding sector is reproduced by the classical dynamics of the spinning strings. The corresponding Yang–Mills operators are the traces of the products of the large number of the elementary fields of the Yang–Mills theory; J corresponds roughly speaking to the number of the elementary fields under the trace. The one-loop anomalous dimension of such operators was computed in [10,11] and the perfect agreement was found with the classical string computations; see the recent review [12] for the details. It turns out that the single trace operators in the $\mathcal{N} = 4$ Yang–Mills theory can be thought of as quantum states of the spin chain, and the one loop anomalous dimension corresponds to the integrable Hamiltonian.

A direct correspondence between the quasiclassical states of the spin chain and the classical string solutions was proposed recently in [13]. It was suggested that in the high energy limit the string worldsheet theory becomes effectively first order in the time derivative and agrees with the Hamiltonian evolution in the spin chain. In our paper we will try to generalize this statement and make it more precise.

The characteristic property of the spinning strings, which was first clearly explained in [14], is that their worldsheets are nearly-degenerate. In all the known situations when there is an agreement with the field theory perturbative computation, every point of the string is moving very fast, approaching the speed of light. Therefore "spinning strings" are actually fast moving strings.[1] This observation suggests that there is a correspondence between a certain class of the Yang–Mills operators and the parameterized null surfaces

[1] Fast moving strings were also considered in this context in [15].

in $AdS_5 \times S^5$ [16]. A null surface is a surface with degenerate metric, ruled by the light rays. A parameterized null surface is a null surface Σ with a function $\sigma : \Sigma \to S^1$ which is constant on the light rays. (On the field theory side σ can be thought of as parameterizing "the position of the elementary field operator inside the trace".) A parameterized null surface can be specified by the embedding functions $x(\sigma, \tau)$ with values in $AdS_5 \times S^5$ such that for a fixed $\sigma = \sigma_0$ the functions $x(\sigma_0, \tau)$ describe a light ray with the affine parameter τ, and $(\partial_\sigma x, \partial_\tau x) = 0$. The embedding functions are defined modulo the "gauge transformations" with the infinitesimal form $\delta x = \phi(\sigma)\partial_\tau x$ where $\phi(\sigma)$ is an arbitrary periodic function of σ.

There is an interesting special case when the null surface is generated by the orbits of the lightlike Killing vector field V in $AdS_5 \times S^5$. The corresponding field theory operators are characterized by a special property that their engineering dimension is equal to a certain combination of conserved charges. In this special case the one loop anomalous dimension should be equal on the string theory side to the value of the conserved charge corresponding to V. We have shown in [16] that this charge is proportional to the following "action functional":

$$S[x] = \int_{S^1} \mathrm{d}\sigma (\partial_\sigma x(\sigma, \tau), \partial_\sigma x(\sigma, \tau)) \tag{1}$$

with the coefficient of the order λ/J^2. (This formula requires a choice of the closed contour on the null surface, but the result of the integration does not actually depend on this choice.) The definition of the special class of operators for which the engineering dimension equals a combination of charges makes sense for finite λ/J^2. What is special about the extremal surfaces corresponding to this particular class of operators for finite λ/J^2? We describe this class of extremal surfaces in Section 2 to the first order in λ/J^2.

We will also generalize the expression (1) for the anomalous dimension for the case when Σ is a general null-surface, not necessarily ruled by the orbits of the symmetry. (The solutions of [17] belong to this more general class.) Following the idea of [13] we will study the long term evolution of the approximating nearly-degenerate extremal surface $\Sigma(\epsilon)$, $\epsilon^2 = \lambda/J^2$, $\Sigma(0) = \Sigma$. We show that this long-term evolution is a Hamiltonian flow on the moduli space of the null-surfaces. The generating function corresponds to the anomalous dimension of the corresponding Yang–Mills operator. The result is a very natural generalization of (1):

$$S[x] = \int_0^{2\pi} \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\sigma (\partial_\sigma x(\sigma, \tau), \partial_\sigma x(\sigma, \tau)) \tag{2}$$

This is a functional on the space of null-surfaces. For its definition it is essential that all the light-like geodesics in $AdS_m \times S^n$ are periodic. Therefore the null-surfaces are also periodic, just like solutions of the massless field equations. The integration over τ corresponds to taking the average over the period, see Section 3 for details. The value of S on the contour should correspond on the field theory side to the anomalous dimension of the corresponding operator.

The structure of the paper. In Section 2 we will study the perturbations of the null surfaces corresponding to the special class of operators for which the engineering dimension is equal to a certain combination of the R-charges. On the AdS side this is reflected in the null-surface $\Sigma(0)$ being invariant under the symmetry generated by the null Killing vector

V. There are restrictions on the nearly-degenerate worldsheet $\Sigma(\epsilon)$ following from the fact that the operators of this class mix only among themselves under the renormgroup. We show that these restrictions can be satisfied.

In Section 3 we will study the perturbations of the null-surfaces which are not generated by the orbits of the light-like Killing vector. We will describe the "long-term" or "secular" behavior of $\Sigma(\epsilon)$. The moduli space of parameterized null-surfaces is a symplectic manifold, and the long-term evolution is a Hamiltonian flow corresponding to the renormgroup flow on the field theory side.

2. Perturbation of the degenerate surfaces ruled by the orbits of the light-like Killing vector

2.1. Summary of this section

Let V be a lightlike Killing vector field in $AdS_5 \times S^5$. Consider the null surfaces which are ruled by the orbits of V. These null surfaces correspond to the Yang–Mills operators of the form tr $F(X, Y, Z)$ where $F(X, Y, Z)$ is some (unsymmetrized) product of X, Y, Z; $X = \Phi_1 + \mathrm{i}\Phi_2$, $Y = \Phi_3 + \mathrm{i}\Phi_4$ and $Z = \Phi_5 + \mathrm{i}\Phi_6$ are the complex combinations of the scalar fields. For these operators the charge corresponding to V is zero in the free theory. Let $\mathcal{O}$ be an operator of this type.

What can we say about the extremal surface $\Sigma_{\mathcal{O}}$ corresponding to such an operator $\mathcal{O}$? We will argue that to the first order in $\epsilon^2 = \lambda/J^2$ the class of extremal surfaces corresponding to this special type of operators can be characterized as follows. For each point $x \in \Sigma_{\mathcal{O}}$ there is a null-surface $\Sigma(0)$ ruled by the orbits of V and such that in the vicinity of x the deviation of $\Sigma_{\mathcal{O}}$ from $\Sigma(0)$ is of the form:

$$x(\tau, \sigma) = x_0(\tau, \sigma) + \epsilon^2 \eta_1(\tau, \sigma) + \cdots, \tag{3}$$

where η_1 has the property:

$$[V, [V, \eta_1]] = 0. \tag{4}$$

Here $[V, \eta_1] = \nabla_V \eta_1 - \nabla_{\eta_1} V$ denotes the commutator of two vector fields; one of these fields is defined only on the surface $\Sigma(0)$, therefore the commutator is also defined only on $\Sigma(0)$. The property $[V, [V, \eta_1]] = 0$ is what characterizes this special class of string worldsheets to the first order in ϵ^2.

Unlike the null-surface $\Sigma(0)$, the nearly-degenerate surface $\Sigma(\epsilon)$ is not invariant under V. But we can describe the variation of $\Sigma(\epsilon)$ under V rather explicitly. Indeed, we can see from (3) that the translation of x by V with the infinitesimal parameter μ is:

$$\mathrm{e}^{\mu V \cdot} x(\tau, \sigma) = x(\tau, \sigma) + \mu\epsilon^2 [V, \eta_1](\tau, \sigma), \tag{5}$$

$$= (x_0(\tau, \sigma) + \mu\epsilon^2 [V, \eta_1](\tau, \sigma)) + \epsilon^2 \eta_1(\tau, \sigma). \tag{6}$$

One can see that when the condition (4) is satisfied, $x_0 + \mu\epsilon^2 [V, \eta_1]$ determines the infinitesimally deformed null-surface. Therefore the translation of the nearly-degenerate extremal surface $\Sigma(\epsilon)$ by V corresponds to the deformation of the approximating null-surface $\Sigma(0)$.

Let us formulate it more precisely. Notice that a light-like Killing vector field V in $AdS_5 \times S^5$ can always be represented as $V = U_A + U_S$ where U_A and U_S are Killing vector fields on AdS_5 and S^5 respectively; $(U_A, U_A) = 1$ and $(U_S, U_S) = -1$. Let Q_S denote the conserved charge corresponding to U_S. Let $\mathcal{N}$ be the moduli space of parameterized null-surfaces ruled by the orbits of V and $\mathcal{M}_J$ be the moduli space of extremal surfaces of the special type characterized by Eq. (4) and such that $Q_S = J$.

Let us choose some map $\Lambda : \mathcal{N} \to \mathcal{M}_J$, such that:

1. For any parameterized null-surface $\Sigma(0)$ the image $\Lambda(\Sigma(0))$ is an extremal surface deviating from $\Sigma(0)$ by the terms of the order $\epsilon^2 = \lambda/J^2$.
2. The density of Q_S on $\Sigma(\epsilon) = \Lambda(\Sigma(0))$ in the limit $\epsilon \to 0$ is proportional to $(1/\epsilon)\,\mathrm{d}\sigma$ where σ is the parameterization of the null-surface $\Sigma(0)$:

$$\text{Density of } Q_S = \frac{\sqrt{\lambda}}{4\pi}\frac{1}{\epsilon}\mathrm{d}\sigma + \mathrm{O}(1) \quad \text{when } \epsilon \to 0. \tag{7}$$

The action of V on $\mathcal{M}_J$ by translations is conjugate by Λ to some one-parameter group of transformations of $\mathcal{N}$. It turns out that this one-parameter group of transformations to the first order in ϵ^2 does not depend on the choice of Λ. It has the following meaning in the dual field theory. We can identify $\mathcal{N}$ with the space of continuous operators in the free field theory. Then the one-parameter group of transformations which we described corresponds to the renormgroup transformations of the continuous operators when we turn on the interaction λ/J^2. This can be summarized in the commutative diagram:

$$\begin{tikzcd} \Sigma(0) \arrow[r, "\text{RG acting on null-surfaces}"] \arrow[d, "\Lambda"'] & \tilde{\Sigma}(0) \arrow[d, "\Lambda"'] \\ \Sigma(\epsilon) \arrow[r, "\text{shift by Killing vector field}"] & \tilde{\Sigma}(\epsilon) \end{tikzcd} \tag{8}$$

In the rest of this section we will explain how to construct the extremal surfaces satisfying the conditions (3) and (4).

2.2. *General facts about the nearly-degenerate surfaces*

Consider the extremal surface in $AdS_5 \times S^5$ which is nearly-degenerate (close to being null). Calculations are simplified with a special choice of the worldsheet coordinates:

$$\left(\frac{\partial x}{\partial \tau}, \frac{\partial x}{\partial \tau}\right) + \epsilon^2 \left(\frac{\partial x}{\partial \sigma}, \frac{\partial x}{\partial \sigma}\right) = 0, \tag{9}$$

$$\left(\frac{\partial x}{\partial \tau}, \frac{\partial x}{\partial \sigma}\right) = 0, \tag{10}$$

where ϵ is a small parameter measuring the deviation of the worldsheet from a null-surface. We assume that σ is periodic with the period 2π. We choose the small parameter ϵ so that the embedding function $x(\tau, \sigma)$ has a finite limit when $\epsilon \to 0$. In this limit $x(\tau, \sigma)$ describes an embedding of the null-surface $x_0(\tau, \sigma)$. If we choose σ as the parameterization of this limiting null-surface then the density of Q_S will agree with this parameterization in the sense of Eq. (7). The string worldsheet action is:

$$S = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{d}\sigma\, \mathrm{d}\tau \left[\frac{1}{\epsilon}(\partial_\tau x, \partial_\tau x) - \epsilon(\partial_\sigma x, \partial_\sigma x) \right]. \tag{11}$$

The string equation of motion is:

$$\frac{1}{\epsilon} D_\tau \partial_\tau x - \epsilon D_\sigma \partial_\sigma x = 0. \tag{12}$$

We denote D_τ and D_σ the worldsheet covariant derivatives. They act on the vector-functions on the worldsheet with values in the tangent space to $AdS_5 \times S^5$. The general definition is

$$D_\tau \xi^\mu = \partial_\tau \xi^\mu + \Gamma^\mu_{\nu\rho} \partial_\tau x^\nu \xi^\rho,$$

where $x^\mu = x^\mu(\tau, \sigma)$ are the coordinate functions specifying the embedding of the string worldsheet into the target space and $\xi^\mu = \xi^\mu(\tau, \sigma)$ is a vector-function on the worldsheet with values in the tangent space $T(AdS_5 \times S^5)$. Somewhat schematically, one can write $D_\tau \xi^\mu = \partial_\tau x^\nu \nabla_\nu \xi^\mu$ where ∇_ν is the covariant derivative in the tangent bundle to the target space. More precisely, D_σ is the natural connection in the 10-dimensional vector bundle over the worldsheet which is the restriction to the worldsheet of the tangent bundle of $AdS_5 \times S^5$. This natural connection is induced from the Levi–Civita connection on $T(AdS_5 \times S^5)$.

One can look for a solution to (12) as a power series in ϵ^2:

$$x(\sigma, \tau) = x_0(\sigma, \tau) + \epsilon^2 \eta_1(\sigma, \tau) + \epsilon^4 \eta_2(\sigma, \tau) + \cdots, \tag{13}$$

where $x_0(\sigma, \tau)$ is a null-surface. The first deviation η_1 satisfies the inhomogeneous Jacobi equation:

$$\frac{D^2 \eta_1}{\partial \tau^2} + R\left(\frac{\partial x_0}{\partial \tau}, \eta_1 \right) \frac{\partial x_0}{\partial \tau} = \frac{D}{\partial \sigma} \frac{\partial x_0}{\partial \sigma} \tag{14}$$

and the constraints:

$$(D_\sigma \eta_1, \partial_\tau x_0) + (D_\tau \eta_1, \partial_\sigma x_0) = 0, \tag{15}$$

$$(D_\tau \eta_1, \partial_\tau x_0) = -\tfrac{1}{2}(\partial_\sigma x_0, \partial_\sigma x_0), \tag{16}$$

where R is the curvature tensor of the target space (we will remind its definition in a moment). The constraints (15) on η follow from the constraints (9) on x. The inhomogeneous Jacobi equation[2] (14) can be derived from the equations of motion (12) in the following way.

[2] The Jacobi equation describes the infinitesimal variation of a geodesic, see for example Appendix 1 of [19]. We decided to keep this name for Eq. (14) which describes the infinitesimal resolution of the null-surface becoming an extremal surface. Indeed, the null-surface is composed of the null-geodesics. After the resolution, these null-geodesics become time-like curves. It is not true that these time-like curves are geodesics, because there is a “riving force” $D_\sigma \partial_\sigma x$ on the right hand side of (14). This driving force, resulting from the tension of the string, makes the equation inhomogeneous.

Consider the family of worldsheets $\Sigma(\epsilon)$ parameterized by $\rho = \epsilon^2$. This family of two-dimensional manifolds "weeps" some three-dimensional manifold (one boundary of this three-dimensional manifold is the null-surface $\Sigma(0)$). Let us think of ρ, σ, τ as coordinates on this three-dimensional manifold. Consider Eq. (12): $D_\tau \partial_\tau x(\rho, \sigma, \tau) - \rho D_\sigma \partial_\sigma x(\rho, \sigma, \tau) = 0$. Differentiate it with respect to ρ:

$$D_\rho D_\tau \partial_\tau x - D_\sigma \partial_\sigma x - \rho D_\rho D_\sigma \partial_\sigma x = 0. \tag{17}$$

Now we have to take into account that the covariant derivatives do not commute. They do not commute because the target space has a non-zero Riemann tensor. To define the Riemann tensor, one takes two vector fields ξ, η and computes the commutator of the covariant derivatives along these two vector fields. The result is a section of $\mathrm{End}(T)(AdS_5 \times S^5)$—the bundle of linear maps from the tangent space to itself. This section is a bilinear function of ξ, η called $R(\xi, \eta)$:

$$R(\xi, \eta) = -\nabla_\xi \nabla_\eta + \nabla_\eta \nabla_\xi + \nabla_{[\xi,\eta]}. \tag{18}$$

For given ξ and η, $R(\xi, \eta)$ is a *matrix* acting in the tangent space to $AdS_5 \times S^5$. The vector fields ∂_ρ, ∂_σ and ∂_τ are defined only on the three-dimensional submanifold. But still, we can compute their commutators and the commutators of the corresponding covariant derivatives. We get, in particular, $[\partial_\rho, \partial_\tau] = 0$ and therefore

$$[D_\rho, D_\tau] = -R(\partial_\rho, \partial_\tau).$$

Let us use this formula in (17). Taking into account also that $D_\rho \partial_\tau x = D_\tau \partial_\rho x$ we get:

$$D_\tau D_\tau \partial_\rho x + R(\partial_\tau x, \partial_\rho x)\partial_\tau x - D_\sigma \partial_\sigma x - \rho D_\rho D_\sigma \partial_\sigma x = 0. \tag{19}$$

In this equation, let us put $\rho = 0$. Since $\partial_\rho x|_{\rho=0} = \eta_1$ we get (14).

Now we will consider the inhomogeneous Jacobi equation in the special case when $\Sigma(0)$ is ruled by the orbits of the light-like Killing vector field. Our aim is to show that in this special case there are solutions satisfying (4).

2.3. A special case of the inhomogeneous Jacobi equation

We will start by rewriting (14) in the special case when $\Sigma(0)$ is ruled by the orbits of V, that is $\partial_\tau x_0 = V(x_0)$:

$$\frac{D^2\eta}{\partial\tau^2} + R(V, \eta)V = \frac{D}{\partial\sigma}\frac{\partial x_0}{\partial\sigma}. \tag{20}$$

Let us introduce an abbreviation for the covariant derivative; for two vector fields α and β we will denote $\alpha \cdot \beta^\mu = \alpha^\nu \nabla_\nu \beta^\mu$. Taking into account (18) we have:

$$D_\tau \eta = V \cdot \eta = [V, \eta] + \eta \cdot V, \tag{21}$$

$$D_\tau^2 \eta = V \cdot (V \cdot \eta) = V \cdot [V, \eta] + [V, \eta] \cdot V - R(V, \eta)V, \tag{22}$$

$$= [V, [V, \eta]] + 2[V, \eta].V - R(V, \eta)V. \tag{23}$$

This allows us to rewrite (20) as:

$$[V,[V,\eta]]+2[V,\eta]\cdot V=\frac{D}{\partial\sigma}\frac{\partial x_0}{\partial\sigma}. \tag{24}$$

Since V is a Killing field, its covariant derivative is antisymmetric: $\nabla_\mu V_\nu=-\nabla_\nu V_\mu$. Therefore for any vector field α we can write $\alpha\cdot V=\iota_\alpha\omega$ where $\omega_{\mu\nu}=\nabla_\mu V_\nu$ is a closed two-form. With this notation Eq. (24) becomes:

$$[V,[V,\eta]]+2\iota_{[V,\eta]}\omega=\frac{D}{\partial\sigma}\frac{\partial x_0}{\partial\sigma}. \tag{25}$$

The null-surfaces ruled by the orbits of the null Killing correspond to operators of the form tr $F(X,Y,Z)$. Consider a degenerate surface $\Sigma(0)$ generated by the orbits of V and its deformation $\Sigma(\epsilon)$ corresponding to turning on the coupling constant. Although $\Sigma(0)$ is invariant under V, its deformation $\Sigma(\epsilon)$ is not invariant. Let us consider the translation of $\Sigma(\epsilon)$ by the vector field V with the parameter μ, schematically $e^{\mu V}\cdot\Sigma(\epsilon)$. This corresponds to the action of the renormgroup on the operator in the theory with a finite coupling constant. The operators of the type tr $F(X,Y,Z)$ are only mixing among themselves under the renormgroup at the level of one loop. This implies that the translation along V of the deformation of the null-surface ruled by the orbits of V should be the deformation of some other null-surface which is also ruled by the orbits of V. For the infinitesimal deformation this means that

$$[V,[V,\eta_1]]=0. \tag{26}$$

Indeed $\mu\epsilon^2[V,\eta_1]$ is the variation of the deformed worldsheet under the shift by $e^{\mu V}\cdot$. Then the condition $[V,[V,\eta_1]]=0$ implies that:

1. $[V,\eta_1]$ is a solution of the *homogeneous* Jacobi equation and therefore $x_0+\mu\epsilon^2[V,\eta_1]$ can be considered as defining the deformed null-surface.[3]
2. This deformed null surface is again ruled by the orbits of V.

Therefore under the condition (26) the shift of $\Sigma(\epsilon)$ by V can be "compensated" by the deformation of $\Sigma(0)$, and the deformed $\Sigma(0)$ is again ruled by the orbits of V. This is precisely the statement that the diagram (8) is commutative, to the first order in ϵ^2.

Can we find η_1 satisfying (25) and (26)? It turns out that we can. Indeed, with the condition (26) Eq. (25) becomes:

$$2\iota_\zeta\omega=\frac{D}{\partial\sigma}\frac{\partial x_0}{\partial\sigma}, \tag{27}$$

where we denoted

$$\zeta=[V,\eta_1].$$

[3] That this deformed surface is degenerate follows from the constraints (15) and (16) and from V being a Killing vector.

We want to study the space of solutions of Eq. (27). The 2-form ω is degenerate, therefore we have to make sure that the right hand side of (27) belongs to the image of ω. To describe the kernel of ω we decompose $V = V_{AdS_5} + V_{S^5}$. Here V_{AdS_5} is the component of V in the tangent space to AdS_5 and V_{S^5} is the component in the tangent space to S^5. The kernel of ω is generated by V and $\tilde{V} = V_{AdS_5} - V_{S^5}$. Notice that $D_\sigma \partial_\sigma x_0$ is orthogonal to V (the proof of this fact uses that V is a Killing vector and V is orthogonal to $\partial_\sigma x_0$). Therefore it is orthogonal to one of the vectors in the kernel of V. It does not follow that $D_\sigma \partial_\sigma x_0$ is orthogonal to $\tilde{V}$. But remember that $\partial_\sigma x_0$ is defined modulo V. Adding to $\partial_\sigma x_0$ something proportional to V we can make it orthogonal to $\tilde{V}$. Indeed, we have

$$(\tilde{V}, D_\sigma \partial_\sigma x_0) = \partial_\sigma (\tilde{V}, \partial_\sigma x_0). \tag{28}$$

and one can change x_0 to $\tilde{x}_0$ where

$$\partial_\sigma \tilde{x}_0 = \partial_\sigma x_0 - \left(\frac{(\tilde{V}, \partial_\sigma x_0) - C}{(\tilde{V}, V)} \right) V, \tag{29}$$

where C is a constant. We adjust C so that $\tilde{x}_0$ is periodic. We have $(\tilde{V}, \partial_\sigma \tilde{x}_0) = C$. Now $(\tilde{V}, D_\sigma \partial_\sigma \tilde{x}_0) = 0$ and therefore $D_\sigma \partial_\sigma \tilde{x}_0$ is orthogonal to the kernel of ω and therefore ω is invertible on it.

We have to also take care of the constraints (15) and (16). Notice that $\zeta = [V, \eta_1]$ is determined from (27) only up to a linear combination of V and $\tilde{V}$. The coefficient of V is undetermined and corresponds to the σ-dependent rescaling of the affine parameter on the light ray. The coefficient of $\tilde{V}$ is fixed to satisfy (16). After that $[V, \eta_1]$ is completely fixed modulo V. It remains to satisfy (15). Let us rewrite (15) in the following form:

$$(V \cdot \eta_1, \partial_\sigma x_0) + (D_\sigma \eta_1, V) = ([V, \eta_1], \partial_\sigma x_0) + (D_\sigma \eta_1, V) - \omega(\partial_\sigma x_0, \eta_1) = 0. \tag{30}$$

We can look for $\eta_1(\tau = 0, \sigma)$ in the form $\eta_1|_{\tau=0} = \alpha(\sigma)\tilde{V} + \beta(\sigma)$ where β is a vector orthogonal to both V and $\tilde{V}$ and α is a function of σ such that:

$$2\partial_\sigma \alpha = -([V, \eta_1], \partial_\sigma x_0) + \omega(\partial_\sigma x_0, \beta).$$

There is a freedom in the choice of β, the only constraint is that α determined from this equation should be a periodic function of σ. This is the freedom to add to η_1 a constant vector $\Delta\eta_1$ (constant means $[V, \Delta\eta_1] = 0$) satisfying $(D_\sigma \Delta\eta_1, V) + (V \cdot \Delta\eta_1, \partial_\sigma x_0) = 0$. This corresponds to the ϵ^2-deformation of the null-surface remaining the null-surface.

The solutions of (25) which have $[V, [V, \eta_1]] \neq 0$ correspond to operators of the form $\mathcal{O} + (\lambda/J^2)\tilde{\mathcal{O}}$ where $\tilde{\mathcal{O}}$ is not annihilated by the symmetry corresponding to V.

2.4. Example: the two-spin solution

Here we will consider as an example the two-spin solution of [18]. This solution is of the type considered in this section, the corresponding null-surface is V-invariant. We will reproduce the terms of the order ϵ^2 in the expansion of the worldsheet near the null-surface.

Let us parameterize the sphere S^5 by the three complex coordinates $Y_I = x_I\, e^{i\phi_I}$ with $\sum_{I=1}^3 x_I^2 = 1$. Of the AdS space we will need only a timelike geodesic, which we parameterize by t. The metric is $-\mathrm{d}t^2 + \sum |\mathrm{d}Y_I|^2$. The lightlike Killing vector is

$$V = \frac{\partial}{\partial t} + \sum_{I=1}^{3} \frac{\partial}{\partial \phi_I}.$$

Consider the following null-surface $x_0^\mu(\sigma, t)$:

$$x_I = x_I(\sigma), \qquad \phi_I(t) = t. \tag{31}$$

The one-form $g_{\mu\nu}V^\nu$ is[4] $V = -\mathrm{d}t + \sum x_I^2\, \mathrm{d}\phi_I$, therefore $\omega = \sum \mathrm{d}x_I^2 \wedge \mathrm{d}\phi_I$. For any vector ξ we have $\iota_\xi \omega = (1/2)\sum[(\xi \cdot x_I^2)\, \mathrm{d}\phi_I - (\xi.\phi_I)\, \mathrm{d}x_I^2]$. The one-form on the right hand side of (27) is:

$$D_\sigma \partial_\sigma x = \sum_I (D_\sigma \partial_\sigma x_I)\, \mathrm{d}x_I. \tag{32}$$

Eq. (27), together with the constraint $(V, [V, \eta]) = -(1/2)(\partial_\sigma x)^2$ can be solved as follows:

$$[V, \eta] = \frac{1}{2}(\partial_\sigma x)^2 \frac{\partial}{\partial t} - \frac{1}{2}\sum_I x_I^{-1} D_\sigma \partial_\sigma x_I \frac{\partial}{\partial \phi_I}. \tag{33}$$

This means, that on the initial surface (31) η is a linear function of t:

$$\eta = t\left[\frac{1}{2}(\partial_\sigma x)^2 \frac{\partial}{\partial t} - \frac{1}{2}\sum_I x_I^{-1} D_\sigma \partial_\sigma x_I \frac{\partial}{\partial \phi_I}\right]. \tag{34}$$

Let us compare this to the solution of [18]. The solutions of [18] correspond to a special finite-dimensional subspace in the space of null-surfaces, such that the contours $x(\sigma, \tau)|_{\tau=\tau_0}$ are the periodic trajectories of the C. Neumann integrable system:

$$D_\sigma \partial_\sigma x_I = -w_I^2 x_I + x_I \sum w_J^2 x_J^2. \tag{35}$$

On such contours,

$$\eta = t\left[\frac{1}{2}\left((\partial_\sigma x)^2 + \sum w_J^2 x_J^2\right)\frac{\partial}{\partial t} + \frac{1}{2}\sum_I w_I^2 \frac{\partial}{\partial \phi_I}\right] \bmod V \tag{36}$$

The expression $\kappa^2 = \sum(\partial_\sigma x_I)^2 + \sum w_J^2 x_J^2$ is twice the energy of the Neumann system. One can see that $x_0 + \epsilon^2 \eta$ gives the zeroth and the first terms in the expansion of the solution of Section 2.1 of [18] around the null-surface.[5]

3. The general case: *V* is not a Killing vector field

In $AdS_5 \times S^5$ the null-geodesics are all periodic with the same period, in a sense that all the light rays emitted from the given point in the future direction will refocus in the future at

[4] We denote the one-form corresponding to the vector V by the same letter; this should not lead to a confusion.
[5] There is a difference in notations: $\kappa^2_{[\mathrm{AFRT}]} = 1 + \epsilon^2\kappa^2$, $w^2_{I[\mathrm{AFRT}]} = 1 + \epsilon^2 w^2$.

some other point. This implies that the null-surfaces in $AdS_5 \times S^5$ are all periodic with the same integer period. The null-surfaces should correspond to the large charge operators at zero coupling; the periodicity of the null-surface corresponds to the fact that the operators in the free theory have zero anomalous dimension.

Turning on a small coupling constant corresponds to considering the extremal surfaces which are very close to being null. Such surfaces are the worldsheets of the "ultra-relativistic" strings. Naively one could think that the extremal surfaces which are close to the null-surfaces are periodic modulo small corrections. But this is not true [13]. It turns out that the worldsheet of the ultrarelativistic string is close to the degenerate surface only locally, in the following sense. For each point on the worldsheet there is a neighborhood with the coordinate size of the order the AdS radius where the surface is indeed close to some null-surface. But as we follow the time evolution the deviation of the extremal surface from the null-surface accumulates in time, and eventually becomes of the order of the radius of the AdS space. This is a manifestation of the general phenomenon which is known in classical mechanics as the "secular evolution" or the "long-term evolution" of the perturbed integrable systems [19]. If the string worldsheet was originally close to a null-surface $\Sigma(0)$ then after evolving for a period of time $\Delta T \sim \epsilon^{-2}$ it will be close to some other null-surface $\Sigma(0)^{(\epsilon^2 \Delta T)}$ which is different from $\Sigma(0)$. Therefore we get a one-parameter family of transformations on the moduli space of the null-surfaces with the parameter ΔT, or rather $\epsilon^2 \Delta T$. We call these transformations the "long term evolution" of the null-surfaces. In fact the fast moving string determines a null-surface and its parameterization, therefore we have a family of transformations on the moduli space of parameterized null-surfaces.

Before we proceed with the analysis of the string, we outline a general situation when this slow evolution is usually found. Suppose that we have an integrable system on the phase space M with the Hamiltonian H_0, and $H_0 + \epsilon^2 \Delta H$ is a perturbed Hamiltonian. We are interested in the special case when the phase space M has a submanifold $M_T \subset M$ closed under the flow of H_0, such that $H_0|_{M_T}$ is constant and all the trajectories of H_0 on M_T are periodic with the same period T. Also, we require that the perturbation is such that the trajectories of $H_0 + \epsilon^2 \Delta H$ which started near M_T will stay near M_T at least on the time intervals $\Delta t \sim \epsilon^{-2}$. In other words, the trajectory of the perturbed Hamiltonian which started on M_T should be always close to some "approximating" periodic trajectory of the unperturbed system. (This does not follow from anywhere; it is an additional assumption which has to be verified.) The "approximating" periodic trajectory will slowly drift. Let us calculate the velocity of the drift. Suppose that we started at the point $x_0 \in M_T$ on the periodic trajectory of H_0 with the period T. Let us denote $x_0(\tau)$ the periodic trajectory of H_0 starting at x_0. The perturbation drives us away from this periodic trajectory. Take n an integer, $n \ll \epsilon^{-2}$. After the time interval nT we are close to the original point x_0. The deviation from x_0 is:

$$\delta x = \epsilon^2 \int_0^{nT} \mathrm{d}\tau (\mathrm{e}^{(nT-\tau)H_0})_* \omega^{-1} \, \mathrm{d}(\Delta H)(x_0(\tau)) + \mathrm{o}(\epsilon^2). \tag{37}$$

Here $(e^{(nT-\tau)H_0})_*$ denotes the translation of the vector in the tangent space to M at the point $x_0(\tau)$ forward to the point $x_0(nT) = x_0$ by the flow of H_0. Let us compute $\iota_{\delta x}\omega$:

$$\iota_{\delta x}\omega = \epsilon^2 \left[\int_0^{nT} d\tau (e^{-(nT-\tau)H_0})^* \, d\Delta H(x_0(\tau)) \right] + o(\epsilon^2). \tag{38}$$

Because of our assumption the component of δx which is transverse to M_T does not accumulate in time. This means that for sufficiently large n we have $(1/n)\delta x$ approximately tangent to $T_{x_0}M_T$ (the component transverse to $T_{x_0}M_T$ is of the order ϵ^2/n.) The one-form on the right hand side of (38) simplifies if we restrict it to the tangent space to M_T. If we take $\xi \in T_{x_0}M_T$ and compute $\omega(\delta x, \xi)$, we will get the difference of $\epsilon^2 \int_0^{nT} \Delta H = n\epsilon^2 \overline{\mathbf{\Delta H}}$ on the periodic trajectory going through $x_0 + \xi$ and the periodic trajectory going through x_0. In this sense,

$$\iota_{\delta x}\omega|_{T_{x_0}M_T} = n\epsilon^2 \, d\overline{\mathbf{\Delta H}}. \tag{39}$$

We have the following picture. Consider the restriction of ω on M_T. Because $H_0|_{M_T} = \text{const}$ the tangent vector to the periodic trajectory is in the kernel of $\omega|_{M_T}$. This means that $\omega|_{M_T}$ defines a closed two-form on the space of periodic trajectories with the period T, which we will denote $\mathbf{\Omega}$. The "averaged" Hamiltonian $\overline{\mathbf{\Delta H}}$ is a function on this space of periodic trajectories. The secular evolution is the vector field $\boldsymbol{\xi}$ on the space of periodic trajectories which satisfies

$$\iota_{\boldsymbol{\xi}}\mathbf{\Omega} = d\overline{\mathbf{\Delta H}}. \tag{40}$$

In the rest of this section we will apply this general scheme to the ultrarelativistic string in $AdS_5 \times S^5$.

3.1. Hamiltonian approach to the fast moving strings

Consider the fast moving string in $AdS_5 \times S^5$. As explained in Section 2.2 of [16] we can parameterize the worldsheet by the coordinates σ and τ such that the embedding functions satisfy the constraints:

$$(\partial_\tau x, \partial_\tau x) + \epsilon^2 (\partial_\sigma x, \partial_\sigma x) = 0, \tag{41}$$

$$(\partial_\tau x, \partial_\sigma x) = 0. \tag{42}$$

These conditions do not completely fix σ and τ. They are preserved by the infinitesimal reparameterizations of the following form:

$$\delta_{(f_L, f_R)} x = [f_L(\sigma + \epsilon\tau) + f_R(\sigma - \epsilon\tau)] \frac{\partial x}{\partial \tau} + \epsilon [f_L(\sigma + \epsilon\tau) - f_R(\sigma - \epsilon\tau)] \frac{\partial x}{\partial \sigma}. \tag{43}$$

We will assume that x is a series in even powers of ϵ: $x = x_0 + \epsilon^2 \eta_1 + \epsilon^4 \eta_2 + \cdots$; this form of x is preserved by the transformations (43) with

$$f_L = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots,$$

$$f_R = f_0 - \epsilon f_1 + \epsilon^2 f_2 - \cdots.$$

Using this residual freedom in the choice of the coordinates we can impose the following condition on the projection of the string worldsheet on S^5:

$$(\partial_\tau x_{S^5}, \partial_\sigma x_{S^5}) = C + \mathrm{O}(\epsilon^2), \tag{44}$$

$$(\partial_\tau x_{S^5}, \partial_\tau x_{S^5}) + \epsilon^2(\partial_\sigma x_{S^5}, \partial_\sigma x_{S^5}) = -1 + \tilde{C}\epsilon^2 + \mathrm{O}(\epsilon^4), \tag{45}$$

where C and $\tilde{C}$ are both constants (do not depend on σ). Rescaling ϵ and τ by $\epsilon^2 \to (1-\tilde{C}\epsilon^2)\epsilon^2$ and $\tau \to (1-\tilde{C}\epsilon^2)^{-1/2}\tau$ we can put

$$\tilde{C} = 0. \tag{46}$$

The initial conditions (44) and (45) are preserved by the equation of motion $D_\tau\partial_\tau x - \epsilon^2 D_\sigma\partial_\sigma x = 0$. This particular choice of the coordinates simplifies the calculations.

In the limit $\epsilon = 0$ the worldsheet of the string becomes a collection of non-interacting massless particles. This limiting system can be described by the action

$$S_0 = \frac{1}{2}\int \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\sigma \left(\frac{\partial x}{\partial \tau}, \frac{\partial x}{\partial \tau}\right), \tag{47}$$

which is the first term of (11). (In this section we will omit the overall coefficient $(\sqrt{\lambda}/4\pi)(1/\epsilon)$ in front of the action.) Introduction of $\epsilon > 0$ corresponds to the perturbation of this system by the interaction between particles, which is described by the second term on the right hand side of (11). The interaction term is

$$\Delta S = \frac{1}{2}\epsilon^2 \int \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\sigma \left(\frac{\partial x}{\partial \sigma}, \frac{\partial x}{\partial \sigma}\right). \tag{48}$$

Let us reformulate this problem in the Hamiltonian approach. We will begin with the study of the unperturbed system (47). Consider first the S^n part. The unperturbed system can be thought of as a continuous family of free non-interacting particles moving on a sphere. For every fixed $\sigma = \sigma_0$, $x(\tau, \sigma_0)$ describes the motion of a free particle which is independent of particles corresponding to other $\sigma \neq \sigma_0$. The momentum conjugate to $x \in S^n$ is $p = \partial x/\partial\tau$, and the Hamiltonian is $H_0 = (1/2)(p, p)$. This system is integrable. For every σ the corresponding point of the string moves on its own geodesic in S^n, different geodesics for different values of σ, and the velocity generally speaking may also depend on σ. The geodesics in S^n are periodic. We can parameterize every geodesic by an angle $\psi \in [0, 2\pi]$. For each σ the "angle" variable $\psi(\sigma)$ satisfies $\partial_\tau\psi(\sigma, \tau) = f(\sigma)$ where $f(\sigma)$ is the σ-dependent frequency. We want to study the effect of the small perturbation (48). Let us first introduce some useful notations.

Particle on a sphere. We will consider two symplectic manifolds. The first is the phase space of a free particle moving on a sphere with the Lagrangian $(\dot{x}, \dot{x})$; we will denote it M. This is the cotangent bundle of the sphere $M = T^*S^n$. The second symplectic manifold is the moduli space of the geodesics in S^n; we will call it G. The natural symplectic form on G can be constructed in the following way. Let us parameterize each geodesic by an angle ψ; we have $(\partial_\psi x, \partial_\psi x) = 1$. The tangent space to the moduli space of geodesics at a given geodesic is given by the Jacobi vector fields ξ which satisfy the Jacobi equation $D_\psi^2\xi - R(\partial_\psi x, \xi)\partial_\psi x = 0$. Given two Jacobi vector fields ξ_1 and ξ_2 we define the symplectic form:

$$\mathbf{\Omega}(\xi_1, \xi_2) = -(\xi_1, D_\psi \xi_2) + (D_\psi \xi_1, \xi_2). \tag{49}$$

The right hand side is evaluated at a particular point on the geodesic (at some particular ψ). But it does not depend on the choice of this point (because of the Jacobi equation). It is closed because it is actually a differential of the one-form $(\partial_\psi x, \xi)$; this one-form does depend on the choice of a point on a geodesic, but its differential does not. Also, a "trivial" Jacobi field $\xi_2 = \partial_\psi x$ corresponding to the shift along the geodesic is in the kernel of $\mathbf{\Omega}$. Indeed,

$$\mathbf{\Omega}(\xi, \partial_\psi x) = (D_\psi \xi, \partial_\psi x) = 0$$

because $(\partial_\psi x, \partial_\psi x) = 1$ for both the original geodesic and its infinitesimal deformation by the Jacobi field ξ. Therefore $\mathbf{\Omega}$ is a well defined two-form on the moduli space of geodesics.

Consider the subspace $M_\times \subset M$ of the phase space where the velocity of the particle is nonzero. It is a fiber bundle over the moduli space of geodesics G. Indeed, the position and the velocity of the particle uniquely determines the geodesic on which the particle is moving. This defines a projection map:

$$\boldsymbol{\pi} : M_\times \to G \tag{50}$$

from the phase space of the particle to the moduli space of geodesics. We will try to use boldface letters to denote objects on G to distinguish them from the functions and forms on M. We decided to use a boldface to denote the projection map because it takes values in G, so $\boldsymbol{\pi}(p, x)$ determines a point in G. The fiber of $\boldsymbol{\pi}$ is $S^1 \times \mathbf{R}_\times$ where $\mathbf{R}_\times$ is a real line without zero. The S^1 parameterizes the position ψ on the geodesic and $\mathbf{R}_\times$ determines the velocity $f = \sqrt{E}$ where we denoted $E = (p, p)$. Let us introduce the 1-form $\mathcal{D}\phi$ on $M_\times$:

$$\mathcal{D}\phi = \frac{(p, \mathrm{d}x)}{(p, p)}. \tag{51}$$

It is characterized by the properties: (1) the restriction of $\mathcal{D}\phi$ on the fiber $S^1 \times \mathbf{R}_\times$ is $E^{-1/2}\,\mathrm{d}\psi$ where ψ is the angle on S^1 and (2) it is zero on any vector in $TM_\times$ having a projection on TS^n orthogonal to p. For a vector $\mathbf{v} \in TG$ we will define a lift $\boldsymbol{\pi}^{-1}\mathbf{v}$ as a vector in $TM_\times$ with $\boldsymbol{\pi}_*(\boldsymbol{\pi}^{-1}\mathbf{v}) = \mathbf{v}$ and $\mathrm{d}E(\boldsymbol{\pi}^{-1}\mathbf{v}) = 0$ and $\mathcal{D}\phi(\boldsymbol{\pi}^{-1}\mathbf{v}) = 0$. This determines the connection on the fiber bundle $M_\times \to G$.

The symplectic form on $M_\times$ can be written in terms of $\mathcal{D}\phi$ and the pull-back of the symplectic form on G:

$$\omega = \tfrac{1}{2}\,\mathrm{d}E \wedge \mathcal{D}\phi + \sqrt{E}\boldsymbol{\pi}^*\mathbf{\Omega}. \tag{52}$$

Particle on $AdS_m \times S^n$. It is straightforward to write the analogue of (52) for the particle moving on AdS_m and on $AdS_m \times S^n$. We consider $AdS_m \times S^n$ with the metric of the mostly negative signature (that is, the metric on S^n is considered negative definite). For two vectors ξ, η in the tangent space to $AdS_m \times S^n$ we denote $(\xi, \eta)_A$ the scalar product of their AdS_m components, and $(\xi, \eta)_S$ the scalar product of their S^n components. In general, the index A will denote objects on AdS_m and the index S objects on the sphere. Let us introduce the notations:

$$E_A = (p, p)_A, \qquad E_S = (p, p)_S, \tag{53}$$

$$\mathcal{D}\phi_A = E_A^{-1}(p, \mathrm{d}x)_A, \qquad \mathcal{D}\phi_S = E_S^{-1}(p, \mathrm{d}x)_S \tag{54}$$

(Notice that E_A is positive and E_S is negative.) We have

$$\boldsymbol{\pi}^*\boldsymbol{\Omega}_A = \frac{(\mathrm{d}p \wedge \mathrm{d}x)_A}{E_A^{1/2}} - \frac{(p, \mathrm{d}p)_A \wedge (p, \mathrm{d}x)_A}{E_A^{3/2}},$$

$$\boldsymbol{\pi}^*\boldsymbol{\Omega}_S = \frac{(\mathrm{d}p \wedge \mathrm{d}x)_S}{(-E_S)^{1/2}} + \frac{(p, \mathrm{d}p)_S \wedge (p, \mathrm{d}x)_S}{(-E_S)^{3/2}}.$$

Therefore

$$\omega = \tfrac{1}{2}\,\mathrm{d}E_A \wedge \mathcal{D}\phi_A + \tfrac{1}{2}\,\mathrm{d}E_S \wedge \mathcal{D}\phi_S + \sqrt{E_A}\boldsymbol{\pi}^*\boldsymbol{\Omega}_A^* + \sqrt{-E_S}\boldsymbol{\pi}^*\boldsymbol{\Omega}_S^*. \tag{55}$$

Here $\boldsymbol{\pi}^*\boldsymbol{\Omega}_A^*$ and $\boldsymbol{\pi}^*\boldsymbol{\Omega}_S^*$ are lifted from the moduli space of geodesics on AdS_m and S^n, respectively; $\mathcal{D}\phi_S = (p, \mathrm{d}x)_S/(p, p)_S$.

String on $AdS_m \times S^n$. Let us proceed with our original system, which is a continuous family of free particles. The phase space of the system is the “loop space” LM which consists of the contours $(p(\sigma), x(\sigma))$ satisfying the constraints $(p, \partial_\sigma x) = 0$ and $(p, p) + \epsilon^2(\partial_\sigma x, \partial_\sigma x) = 0$. The symplectic form is an integral over σ:

$$\omega = \int \mathrm{d}\sigma \left[\frac{1}{2}\,\mathrm{d}E_A(\sigma) \wedge \mathcal{D}\phi_A(\sigma) + \frac{1}{2}\,\mathrm{d}E_S(\sigma) \wedge \mathcal{D}\phi_S(\sigma) \right. \\ \left. + \sqrt{E_A(\sigma)}\boldsymbol{\pi}^*\boldsymbol{\Omega}_A^*(\sigma) + \sqrt{-E_S(\sigma)}\boldsymbol{\pi}^*\boldsymbol{\Omega}_S^*(\sigma)\right]. \tag{56}$$

We want to derive an evolution equation on LG. We use the boldface for the objects living on G or LG, therefore our goal is to arrive at the equation where all the letters are bold. The differential of the perturbation Hamiltonian is

$$\mathrm{d}\Delta H = \int \mathrm{d}\sigma(\partial_\sigma x, D_\sigma\,\mathrm{d}x) = -\int \mathrm{d}\sigma(D_\sigma\partial_\sigma x, \mathrm{d}x).$$

Let us decompose $\mathrm{d}x$ as the sum of the component parallel to $p = \partial_\tau x$ and the component orthogonal to p. We get:

$$\mathrm{d}\Delta H = \int \mathrm{d}\sigma[-(p(\sigma), D_\sigma\partial_\sigma x)_A\mathcal{D}\phi_A(\sigma) - (p(\sigma), D_\sigma\partial_\sigma x)_S\mathcal{D}\phi_S(\sigma) \\ - (\mathrm{d}x(\sigma), (D_\sigma\partial_\sigma x)_\perp)]. \tag{57}$$

Here $(D_\sigma\partial_\sigma x)_\perp = D_\sigma\partial_\sigma x - [(p, D_\sigma\partial_\sigma x)_A/(p, p)_A]p_A - [(p, D_\sigma\partial_\sigma x)_S/(p, p)_S]p_S$. The one-form $(\mathrm{d}x, (D_\sigma\partial_\sigma x)_\perp)$ is an element of the cotangent space $T^*_{(p,x)}M$ to the phase space at the point (p, x). It is horizontal in the sense that its value on $\partial/\partial E_A$, $\partial/\partial E_S$, $\partial/\partial\phi_A$ and

$\partial/\partial\phi_S$ is zero. This means that it is a pullback of some form $\alpha(p, x)$ on the tangent space to G at the point $\boldsymbol{\pi}(p, x)$:

$$(\mathrm{d}x, (D_\sigma\partial_\sigma x)_\perp) = \boldsymbol{\pi}^*\alpha(p, x). \tag{58}$$

To avoid confusion, we want to stress that this form $\alpha(p, x) \in T^*_{\pi(p,x)}G$ depends on (p, x) and not just on the projection $\boldsymbol{\pi}(p, x)$. That is why we did not use the boldface for α. Given Eq. (58) for $\mathrm{d}H$ and the symplectic form (56) on LM we can write down the Hamiltonian vector field:

$$\omega^{-1}\,\mathrm{d}(H + \epsilon^2\Delta H) = \frac{\partial}{\partial\phi_A} + \frac{\partial}{\partial\phi_S} + \epsilon^2\left[(p, D_\sigma\partial_\sigma x)_A\frac{\partial}{\partial E_A} + (p, D_\sigma\partial_\sigma x)_S\frac{\partial}{\partial E_S} - \boldsymbol{\pi}^{-1}\boldsymbol{\Omega}^{-1}\alpha(p, x)\right]. \tag{59}$$

Long term evolution. The coefficients of $\partial/\partial E_A$ and $\partial/\partial E_S$ describe the evolution of the frequency:

$$E_A(\tau) = E_A(0) + \epsilon^2\int_0^\tau \mathrm{d}\tau'(p(\sigma, \tau'), D_\sigma\partial_\sigma x(\sigma, \tau'))_A,$$

$$E_S(\tau) = E_S(0) + \epsilon^2\int_0^\tau \mathrm{d}\tau'(p(\sigma, \tau'), D_\sigma\partial_\sigma x(\sigma, \tau'))_S.$$

We want to study the evolution over the period up to the order ϵ^2 therefore we can replace on the right hand side $x(\sigma, \tau')$ and $p(\sigma, \tau')$ with the unperturbed motion $x_0(\sigma, \tau')$ and $p_0(\sigma, \tau')$.

We can now see that $E_A(\tau)$ and $E_B(\tau)$ oscillates around $E_A(0)$ and $E_B(0)$. Indeed, taking into account the initial condition (44) we have:

$$\int \mathrm{d}\tau'(\partial_{\tau'}x(\sigma, \tau'), D_\sigma\partial_\sigma x(\sigma, \tau'))_A = -\frac{1}{2}\int \mathrm{d}\tau'\frac{\partial}{\partial\tau'}(\partial_\sigma x(\sigma, \tau'), \partial_\sigma x(\sigma, \tau'))_A = 0 \tag{60}$$

because of the periodicity. Therefore the variations of the frequency do not accumulate over time. The initial conditions (45) imply that $E_A(0) = 1 - \epsilon^2(\partial_\sigma x, \partial_\sigma x)_A +$ (terms of the higher order in ϵ^2).

But the variation of the shape of the contour does accumulate. For τ of the order $1/\epsilon^2$ the change in the shape of the contour will be of the order 1. Indeed (59) implies that the projection of the trajectory on G satisfies:

$$\partial_\tau\boldsymbol{\pi}(p, x) = -\epsilon^2\boldsymbol{\Omega}^{-1}\alpha(p, x) \tag{61}$$

The variation of the geodesic over one period is therefore:

$$\delta\boldsymbol{\pi} = -\boldsymbol{\Omega}^{-1}\int_0^{2\pi}\mathrm{d}\psi\,\alpha(p_0, x_0(\psi)). \tag{62}$$

Again, we neglected the higher order terms in ϵ^2 and replaced all the $(p(\tau), x(\tau))$ on the right hand side of (61) by the unperturbed $p_0(\tau), x_0(\tau)$. Also, following the notations in (49) we replaced the time τ with the angle ψ parameterizing the geodesic. Notice that

$\int_0^{2\pi} \mathrm{d}\psi\, \alpha(p_0, x_0(\psi))$ is the differential of the function on the base G which is obtained by the integration of ΔH over ψ:

$$\int_0^{2\pi} \mathrm{d}\psi\, \alpha = \mathrm{d}\overline{\mathbf{\Delta H}}, \tag{63}$$

$$\overline{\mathbf{\Delta H}} = \frac{1}{2}\int_0^{2\pi} \mathrm{d}\psi \int_0^{2\pi} \mathrm{d}\sigma(\partial_\sigma x_0(\psi,\sigma), \partial_\sigma x_0(\psi,\sigma)). \tag{64}$$

Let us prove it. We have

$$\int \mathrm{d}\psi\, \alpha = \int \mathrm{d}\psi\, \mathrm{d}\sigma(D_\sigma\, \mathrm{d}x_\perp(\psi,\sigma), \partial_\sigma x(\psi,\sigma)) \tag{65}$$

By definition $\mathrm{d}x_\perp = \mathrm{d}x - (\mathrm{d}x, \partial_\psi x)_A \partial_\psi x_A + (\mathrm{d}x, \partial_\psi x)_S \partial_\psi x_S$. (Remember that in our notations the metric on S^5 is negative definite.) Therefore:

$$\int \mathrm{d}\psi\, \alpha = \int \mathrm{d}\psi\, \mathrm{d}\sigma(D_\sigma\, \mathrm{d}x, \partial_\sigma x) - \int \mathrm{d}\psi\, \mathrm{d}\sigma(D_\sigma((\partial_\psi x, \mathrm{d}x)_A \partial_\psi x), \partial_\sigma x)_A$$
$$- \int \mathrm{d}\psi\, \mathrm{d}\sigma(D_\sigma((\partial_\psi x, \mathrm{d}x)_S \partial_\psi x), \partial_\sigma x)_S.$$

But the second and the third terms on the right hand side are zero on the initial conditions (44). Therefore $\int \mathrm{d}\psi\, \alpha = \mathrm{d}\overline{\mathbf{\Delta H}}$ as we wanted.

Now we can compute the variation of $\boldsymbol{\pi}(p, x)$ over the period:

$$\delta\boldsymbol{\pi} = -\epsilon^2 \mathbf{\Omega}^{-1}\, \mathrm{d}\overline{\mathbf{\Delta H}}(\boldsymbol{\pi}). \tag{66}$$

Introducing $\mathbf{t} = \epsilon^2\tau$ we obtain the equation for the secular evolution:

$$\frac{\partial\boldsymbol{\pi}}{\partial\mathbf{t}} = -\mathbf{\Omega}^{-1}\, \mathrm{d}\overline{\mathbf{\Delta H}}(\boldsymbol{\pi}). \tag{67}$$

In this equation all the letters (except for d and ∂) are boldface, as we wanted. It describes the evolution of the contour in the moduli space of null-geodesics on $AdS_m \times S^n$.

3.2. Summary

The effective Hamiltonian is a functional on the space of parameterized null-surfaces:

$$\overline{\mathbf{\Delta H}} = \frac{1}{2}\int_0^{2\pi} \mathrm{d}\psi \int_0^{2\pi} \mathrm{d}\sigma(\partial_\sigma x, \partial_\sigma x). \tag{68}$$

Here ψ is the affine parameter on the light rays and that the periodicity of the light rays is $\Delta\psi = 2\pi$. The remaining coordinate freedom is in the choice of the closed contour $\psi =$ const., but the integral (68) does not depend on this choice. Therefore it is a functional on the space of parameterized null surfaces.

The symplectic form on the space of parameterized null-surfaces is

$$\mathbf{\Omega} = \int \mathrm{d}\sigma(\mathrm{d}x \wedge D_\psi\, \mathrm{d}x). \tag{69}$$

This symplectic form has a straightforward geometrical interpretation. Notice that the space of classical string worldsheets has a natural symplectic form which is defined in the following way. The deformations of the string worldsheet are described by the vector fields $\xi(\sigma, \tau)$. The value of the symplectic form on two infinitesimal deformations ξ_1 and ξ_2 is

$$\Omega_{\text{string}}(\xi_1, \xi_2) = \frac{\sqrt{\lambda}}{2\pi} \oint ((\xi_1, *D\xi_2) - (\xi_2, *D\xi_1)). \tag{70}$$

Here D is the covariant differential on the worldsheet, the metric on the worldsheet is induced from the space–time, the integral is taken over a closed spacial contour and the fields ξ_1 and ξ_2 are chosen to preserve the conformal structure on the worldsheet (they are originally defined only up to the vector tangent to the worldsheet). The symplectic form (69) on the space of null-surfaces is the ultrarelativistic limit of the symplectic form (70) on the phase space of the classical string. Indeed, when $\epsilon \to 0$ (70) becomes

$$\Omega_{\text{string}} = \frac{\sqrt{\lambda}}{2\pi\epsilon} \int (\mathrm{d}x \wedge D_\psi\, \mathrm{d}x). \tag{71}$$

As we will explain in Section 3.4, this equation justifies our definition of the small parameter ϵ and the parameterization σ. Indeed, the right hand side agrees on the field theory side with the symplectic structure of the continuous limit of the spin chain. The parameter σ should be identified with the number of the site divided by the length of the chain.

In the end of this section we will derive this evolution equation (67) directly from the inhomogeneous Jacobi equation. But first we want to rewrite (67) in a more explicit form and discuss its interpretation in the dual gauge theory.

3.3. *Explicit evolution equations*

Here we will realize the moduli space of geodesics as a quadric in the complex projective space and write the evolution Eq. (67) in the explicit form. Let us start with the S^n part. Geodesics on S^n are equators:

$$x_0(\tau, \sigma) = e_1(\sigma) \cos\tau + e_2(\sigma) \sin\tau. \tag{72}$$

They are parameterized by a pair of orthogonal vectors e_1 and e_2 modulo the orthogonal transformations mixing e_1 and e_2. As a manifold it is the Grassmanian of two-dimensional planes in the $n+1$-dimensional space, $G = Gr(2, n+1)$. Let us introduce a complex vector $Z = e_1 + \mathrm{i}e_2$ in $\mathbf{C}^{n+1}$. It has the properties $(Z, Z) = 0$ and $(\bar{Z}, Z) = 2$. Given the equator, Z is determined up to a phase $Z \to \mathrm{e}^{\mathrm{i}\alpha} Z$. Therefore the moduli space of geodesics is a quadric in the complex projective space $\mathbf{CP}^n$ given in the homogeneous coordinates $[Z_1 : \cdots : Z_{n+1}]$ by the equation $(Z, Z) = 0$. Similarly, the moduli space of geodesics on AdS_m is a quadric in $\mathbf{CP}^m$ given in the homogeneous coordinates $[Y_{-1}, Y_0, \ldots, Y_{m-1}]$ by the equation $(Y, Y) = Y_{-1}^2 + Y_0^2 - Y_1^2 - \cdots - Y_{m-1}^2 = 0$.

In our application we need actually not just the geodesic, but also the position of the point on it. Therefore we have to keep the phases of Z and Y. The position of the point of the string in $AdS_m \times S^n$ is given by

$$(x_A, x_S) = (\mathrm{Re}\, Y, \mathrm{Re}\, Z)$$

and the velocity is

$$(p_A, p_S) = (\sqrt{E_A}\,\mathrm{Im}\,Y, \sqrt{-E_S}\,\mathrm{Im}\,Z).$$

The averaged perturbation Hamiltonian is

$$\overline{\mathbf{\Delta H}} = \frac{1}{4}\int \mathrm{d}\sigma[(\partial_\sigma \overline{Y}, \partial_\sigma Y) - (\partial_\sigma \overline{Z}, \partial_\sigma Z)] \tag{73}$$

with the constraint

$$(\overline{Y}, \partial_\sigma Y) - (\overline{Z}, \partial_\sigma Z) = 0. \tag{74}$$

The symplectic form

$$\Omega = \frac{1}{2\mathrm{i}}\int \mathrm{d}\sigma((\mathrm{d}\overline{Y} \wedge \mathrm{d}Y) - (\mathrm{d}\overline{Z} \wedge \mathrm{d}Z)). \tag{75}$$

The Hamiltonian flow (59) *averaged over the period* is:

$$\begin{aligned} \partial_\tau Y &= \mathrm{i}[(1 - \tfrac{1}{2}\epsilon^2(\partial_\sigma \overline{Y}, \partial_\sigma Y))Y - \tfrac{1}{2}\epsilon^2\partial_\sigma^2 Y - \tfrac{1}{4}\epsilon^2(\partial_\sigma Y, \partial_\sigma Y)\overline{Y}],\\ \partial_\tau Z &= \mathrm{i}[(1 - \tfrac{1}{2}\epsilon^2(\partial_\sigma \overline{Z}, \partial_\sigma Z))Z - \tfrac{1}{2}\epsilon^2\partial_\sigma^2 Z - \tfrac{1}{4}\epsilon^2(\partial_\sigma Z, \partial_\sigma Z)\overline{Z}]. \end{aligned} \tag{76}$$

The terms proportional to Y and Z are fixed from the initial condition (45), and the terms proportional to $\overline{Y}$ and $\overline{Z}$ are such that $(\partial_\tau Y, Y) = 0$ and $(\partial_\tau Z, Z) = 0$.

3.4. Interpretation in the dual field theory

To interpret these equations on the field theory side we have to consider the single trace operators with large R-charge. In the "continuum limit" Z corresponds to the local density of the R charge. The operators corresponding to the speeding strings are "locally half-BPS" [14]. Therefore the density of the R charge should be a decomposable element of so(6) which means that $(Z, Z) = 0$. Following the idea of [13] we can interpret Z as parameterizing a point on the coadjoint orbit of so(6) consisting of the decomposable elements. Decomposable elements are those antisymmetric matrices which can be represented as an antisymmetric product of two orthogonal vectors $e_1 \wedge e_2$; then $Z = e_1 + \mathrm{i}e_2$. This orbit corresponds in the sense of [20] to the vector representation of so(6) which lives on the sites of the spin chain.

Let us now add the AdS part. Consider the orbit of so(2, 4) consisting of the elements of the form $Y \wedge \overline{Y}$ where $Y = \tilde{e}_1 + \mathrm{i}\tilde{e}_2$ with $(Y, Y) = 0$ and $(\overline{Y}, Y) = 2$. Just as a geodesic in S^5 is defined by Z modulo a phase, a geodesic in AdS_5 is defined by Y modulo a phase. Roughly speaking, a pair of functions $(Z(\sigma), Y(\sigma))$ where both $Z(\sigma)$ and $Y(\sigma)$ are defined modulo local phase rotations (independent for Z and Y) define a null-surface in $AdS_5 \times S^5$. But there is a subtlety. For the corresponding surface to be null we have to be able to fix the relative phase of Y and Z in such a way that

$$(\overline{Z}, \partial_\sigma Z) = (\overline{Y}, \partial_\sigma Y). \tag{77}$$

This imposes the following integrality condition on the functions $Y(\sigma)$ and $Z(\sigma)$. Let us consider a two-dimensional surface D_Z in $\mathbf{CP}^6$ such that its boundary is the contour $[Z(\sigma)]$

and a two-dimensional surface D_Y in $\mathbf{CP}^{2+4}$ such that its boundary is the contour $[Y(\sigma)]$. The integrality condition is that the symplectic area of D_Y should be equal to the symplectic area of D_Z plus an integer. On the field theory side this integrality condition corresponds to the cyclic invariance of the trace.

To summarize, let us consider two functions $[Y] : S^1 \to \mathbf{CP}^{2+4}$ and $[Z] : S^1 \to \mathbf{CP}^6$ satisfying $(Y, Y) = (Z, Z) = 0$ and the integrality condition described above. The integrality condition guarantees that we can lift $[Z]$ and $[Y]$ to the functions $Y : S^1 \to \mathbf{C}^{2+4}$ and $Z : S^1 \to \mathbf{C}^6$ satisfying (77). Let us fix such a lift modulo an overall phase $(Y, Z) \sim \mathrm{e}^{\mathrm{i}\phi(\sigma)}(Y, Z)$. This data determines the null surface in $AdS_5 \times S^5$ corresponding to the Yang–Mills operator with the anomalous dimension

$$\epsilon\frac{\sqrt{\lambda}}{8\pi}\int \mathrm{d}\sigma((\partial_\sigma \bar{Z}, \partial_\sigma Z) - (\partial_\sigma \bar{Y}, \partial_\sigma Y)). \tag{78}$$

In this formula we have restored the coefficient $\sqrt{\lambda}/4\pi\epsilon$ from Eq. (11). The integral does not depend on the “overall” phase of (Y, Z).

The precise relation between ϵ and λ can be obtained by computing the conserved charges. Consider a Killing vector field U on S^5. We have

$$\delta_U x^i = u^{ij} x^j, \tag{79}$$

where x^i, $i = 1, \ldots, 6$ denote a unit vector representing the point of S^5 and u^{ij} is an antisymmetric matrix corresponding to the symmetry U. Let us compute the corresponding conserved charge to the first order in ϵ. We have:

$$Q_U = \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\, u^{ij} x_0^j(\tau, \sigma)\partial_\tau x_0^i(\tau, \sigma). \tag{80}$$

By definition $x_0(\tau, \sigma)$ should belong to the geodesic specified by $Z(\sigma)$, and $\partial_\tau x^i = ((\mathrm{i}/2)Z \wedge \bar{Z})^{ij} x^j$. This means that the charge is:

$$Q_U = \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma \left(u, \frac{\mathrm{i}}{2}\bar{Z} \wedge Z\right). \tag{81}$$

But $(\mathrm{i}/2)\bar{Z} \wedge Z$ should be the local density of the R charge. Therefore we identify

$$\epsilon = \frac{\sqrt{\lambda}}{2\pi(L/2\pi)}, \tag{82}$$

where L is the length of the spin chain (the number of operators under the trace.) Substitution of ϵ in (78) gives:

$$\Delta = \frac{1}{16\pi^2}\frac{\lambda}{L/2\pi}\int_0^{2\pi} \mathrm{d}\sigma((\partial_\sigma \bar{Z}, \partial_\sigma Z) - (\partial_\sigma \bar{Y}, \partial_\sigma Y)). \tag{83}$$

This is a functional on the space of contours $(Y(\sigma), Z(\sigma))$ in $\mathbf{C}^{12}$, subject to the constraints $|Y|^2 = |Z|^2 = 2$ and $(\bar{Z}, \partial_\sigma Z) = (\bar{Y}, \partial_\sigma Y)$ and defined up to an overall phase $(Y(\sigma), Z(\sigma)) \to \mathrm{e}^{\mathrm{i}\phi(\sigma)}(Y(\sigma), Z(\sigma))$. The symplectic structure on this space is given in (75).

3.5. Comment on the special case when $\Sigma(0)$ is generated by the orbits of V

In the special case when $\Sigma(0)$ is generated by the orbits of V the anomalous dimension can be computed in two different ways. One way is to compute the conserved charge corresponding to V as was done in [7]. The other way suggested in [13] is to study the secular evolution of $\Sigma(\epsilon)$ and find the Hamiltonian governing this evolution. The two methods give the same result for the following reason. The constraint $(\partial_\tau x)^2 + \epsilon^2(\partial_\sigma x)^2 = 0$ says that the total perturbed Hamiltonian $H_0 + \epsilon^2 \Delta H$ should be zero. The "effective" Hamiltonian governing the secular drift is obtained by the averaging of ΔH over the period. Because of the constraint we have $\epsilon^2 \Delta H = -H_0$. But in the vicinity of $\Sigma(0)$ we have H_0 equal to the charge Q_V up to the terms of the higher order in the deviation from $\Sigma(0)$. (This follows from the fact that the Hamiltonian flow of H_0 on $\Sigma(0)$ is the translation by V.)

3.6. Direct derivation from the Jacobi equation

We derived (67) and (76) using the Hamiltonian formalism. Here we will give a direct derivation from the inhomogeneous Jacobi equation.

Let us study the inhomogeneous Jacobi equation in the special case of AdS times a sphere:

$$D_\tau^2 \eta - R(\partial_\tau x, \eta)\partial_\tau x = D_\sigma \partial_\sigma x. \tag{84}$$

We can decompose $\partial_\tau x$ as a sum of the vector $\partial_\tau x_{AdS_5}$ in the tangent space to AdS_5 and the vector $\partial_\tau x_{S^5}$ in the tangent space to S^5, $\partial_\tau x = \partial_\tau x_{AdS_5} + \partial_\tau x_{S^5}$. We denote $\widetilde{\partial_\tau x} = \partial_\tau x_{AdS_5} - \partial_\tau x_{S^5}$. We will need the following representation for $D_\sigma \partial_\sigma x$:

$$D_\sigma \partial_\sigma x = D_\tau \xi + \alpha(\sigma, \tau)\partial_\tau x + \beta(\sigma, \tau)\widetilde{\partial_\tau x}, \tag{85}$$

where ξ is a Jacobi field orthogonal to both $\partial_\tau x$ and $\widetilde{\partial_\tau x}$ and $\alpha(\tau)$ and $\beta(\tau)$ are some functions. Indeed, let us consider the projection of the geodesic on S^5. The geodesic on S^5 is an equator:

$$x(\tau, \sigma) = e_1(\sigma)\cos\tau + e_2(\sigma)\sin\tau, \tag{86}$$

where $(e_1(\sigma), e_1(\sigma)) = (e_2(\sigma), e_2(\sigma)) = 1$ and $(e_1(\sigma), e_2(\sigma)) = 0$. We have

$$D_\sigma \partial_\sigma x = (e_1''(\sigma)\cos\tau + e_2''(\sigma)\sin\tau)_{\|}, \tag{87}$$

where the index $||$ means that we have to project to the tangent space of S^5 along the radial direction. It is enough to consider this equation at $\sigma = 0$. Let us decompose the second derivative of e_i, $i = 1, 2$ in the components $a_{i,\mathrm{tang}}$ and $a_{i,\mathrm{norm}}$ parallel to the plane (e_1, e_2) and the components $(e_i'')_{\mathrm{vert}}$ perpendicular to this plane:

$$e_1'' = a_{1,t}e_2 + a_{1,n}e_1 + (e_1'')_{\mathrm{vert}}, \tag{88}$$

$$e_2'' = a_{2,t}e_1 + a_{2,n}e_2 + (e_2'')_{\mathrm{vert}}. \tag{89}$$

The second covariant derivative is:

$$\begin{aligned} D_\sigma \partial_\sigma x(\tau, \sigma) = {} & (a_{1,t}\cos^2\tau - a_{2,t}\sin^2\tau + (a_{2,n} - a_{1,n})\cos\tau\sin\tau) \\ & \times \partial_\tau(e_1\cos\tau + e_2\sin\tau) + (e_1'')_{\mathrm{vert}}\cos\tau + (e_2'')_{\mathrm{vert}}\sin\tau. \end{aligned} \tag{90}$$

The analogous expression holds for the AdS_5-component of $D_\sigma \partial_\sigma x$. But $(e_1'')_{\mathrm{vert}} \cos\tau + (e_2'')_{\mathrm{vert}} \sin\tau = \partial_\tau((e_1'')_{\mathrm{vert}} \sin\tau - (e_2'')_{\mathrm{vert}} \cos\tau)$ and

$$\xi = (e_1'')_{\mathrm{vert}} \sin\tau - (e_2'')_{\mathrm{vert}} \cos\tau$$

is a Jacobi field. This proves (85). Notice that ξ and $D_\tau \xi$ are orthogonal to both $\partial_\tau x$ and $\widetilde{\partial_\tau x}$. We can now present a solution to the equation (84):

$$\eta = \tfrac{1}{2}\tau\xi + A\partial_\tau x + B\widetilde{\partial_\tau x}, \tag{91}$$

where A and B satisfy $\partial^2 A/\partial\tau^2 = \alpha$ and $\partial^2 B/\partial\tau^2 = \beta$. It is important that both A and B can be chosen periodic functions of τ. This is true for B:

$$\int \mathrm{d}\tau\, \beta = \int \mathrm{d}\tau (\partial_\tau x, D_\sigma \partial_\sigma x) = -\frac{1}{2} \int \mathrm{d}\tau \partial_\tau (\partial_\sigma x, \partial_\sigma x) = 0 \tag{92}$$

and also for A, because

$$\int \mathrm{d}\tau\, \alpha = \int \mathrm{d}\tau (\widetilde{\partial_\tau x}, D_\sigma \partial_\sigma x) \tag{93}$$

$$= -\frac{1}{2} \int \mathrm{d}\tau \partial_\tau [(\partial_\sigma x, \partial_\sigma x)_{AdS_5} - (\partial_\sigma x, \partial_\sigma x)_{S^5}] = 0 \tag{94}$$

since the projections of x to AdS_5 and S^5 are both periodic. Therefore we see that η can be chosen as a sum of the linearly growing term and the oscillating terms. The linearly growing term is $(1/2)t\xi$ where ξ is a Jacobi field satisfying $D_\tau \xi = D_\sigma \partial_\sigma x$. This linear term is responsible for the secular evolution.

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